ON VARIATIONAL PRINCIPLES OF THERMOELASTICITY

(O VARIATSIONNYKH URAVNENIIAKH TERMOUPRUGOSTI)

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Variational principles for the thermoelastic problem with heat sources and sinks are deduced. In the absence of sources and sinks the analogous variational equation was obtained by Biot [1], on the basis of thermodynamics of linear irreversible processes. It is shown under what conditions Biot's generalized variational equation passes to the variational equations of thermodynamics of equilibrium processes [2,3].

At the instant of time $\tau = 0$, let an elastic body have a constant absolute temperature T and let it be in its natural state, i.e. when the stresses and strains are absent.

At the instant of time τ, due to the effect of external forces and temperatures, given boundary conditions on the surface, as well as due to internal heat sources and sinks, the strains and temperatures inside the body will take on the values

$$e_{ik} = e_{ik}(x_k, \tau), \quad T_1 = T + \theta \qquad (\theta = \theta(x_k, \tau))$$

where $x_k (k = 1, 2, 3)$ are Cartesian coordinates.

We shall assume that the Duhamel-Neumann equations are valid at any instant of time

$$\sigma_{ik} = 2\mu e_{ik} + (\lambda e - \beta\theta)\,\delta_{ik}, \qquad \delta_{ik} = \begin{cases} 1 & \text{for } i = k \\ 0 & \text{for } i \neq k \end{cases} \tag{1}$$

Here e_{ik} are the components of the strain tensor

$$e_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right) \qquad \left(\beta = \frac{E\alpha}{1-2\nu}\right) \tag{2}$$

α is the coefficient of linear expansion, ν is Poisson's ratio, λ and μ are Lame's constants. In the absence of body forces the equations of equilibrium in terms of displacements and the boundary conditions in terms of tractions are of the form

$$\mu\nabla^2\mathbf{u} + (\lambda + \mu)\,\mathrm{grad}\, e - \beta\,\mathrm{grad}\,\theta = 0, \qquad \sigma_{ik}\, l_k - P_i = 0 \tag{3}$$

where $\mathbf{u}$ is the displacement vector, $e = \mathrm{div}\,\mathbf{u}$, P_i are the components of the surface traction vector, l_k are the direction cosines of the external normal.

To obtain the equation which relates the deformations to the temperature, we use the equation of conservation of heat $dh = -\,\mathrm{div}\,\mathbf{q}\,d\tau + dw$, and defining the heat-flux vector $\mathbf{q}$ by the temperature gradient in accordance with Fourier's law $\mathbf{q} = -\,k\,\mathrm{grad}\,\theta$, we obtain

$$dh = k\nabla^2\theta d\tau + dw \tag{4}$$

Here $dw/d\tau$ is the specific strength of the heat sources and sinks, k is the coefficient of heat conduction.

The quantity of heat dh, received by an element of volume within the time interval $d\tau$, may be calculated if the density of the internal energy of the body is known. Since the internal energy is a function of state, it may be assumed in calculating it that the passage from the natural equilibrium state to any non-equilibrium state, corresponding to the instant of time τ, is realized in a reversible manner.

Let us introduce some generalized heat capacity C_{ik}. Then we obtain for the increment of the internal energy density

$$d\ni = dh + \sigma_{ik}de_{ik} = (\sigma_{ik} + C_{ik})\, de_{ik} + Cd\theta \tag{5}$$

Here C is the heat capacity for constant volume.

From the definition of internal energy as a function of state, and on the basis of (5), we have

$$\frac{\partial \ni}{\partial e_{ik}} = \sigma_{ik} + C_{ik}, \qquad \frac{\partial \ni}{\partial \theta} = C \tag{6}$$

We calculate the increment of the entropy density per unit of volume

$$ds = \frac{dh}{T_1} = \frac{C_{ik}}{T_1}\, de_{ik} + \frac{C}{T_1}\, d\theta \tag{7}$$

From the second law of thermodynamics it follows that ds is a total differential of the independent variables e_{ik} and θ, i.e.

$$\frac{\partial s}{\partial e_{ik}} = \frac{C_{ik}}{T_1}, \qquad \frac{\partial s}{\partial \theta} = \frac{C}{T_1} \tag{8}$$

Eliminating from Equations (6) and (8) the internal energy density and the entropy, respectively, and using Equation (1) we obtain

$$\frac{\partial C}{\partial e_{ik}} = -\beta\delta_{ik} + \frac{\partial C_{ik}}{\partial \theta}, \quad \frac{\partial C}{\partial e_{ik}} = -\frac{C_{ik}}{T_1} + \frac{\partial C_{ik}}{\partial \theta} \tag{9}$$

It follows

$$C_{ik} = \beta T_1 \delta_{ik}, \qquad C = C(\theta) \tag{10}$$

Having the expression for the heat capacity (10) we find in accordance with (6)

$$\frac{\partial \mathfrak{s}}{\partial e_{ik}} = 2\mu e_{ik} + (\lambda e + \beta T)\,\delta_{ik}, \quad \frac{\partial \mathfrak{s}}{\partial \theta} = C \tag{11}$$

Assuming the quantity θ to be small as compared to the initial absolute temperature of the body T, and assuming the heat capacity C, as well as the constants λ, μ and β to be independent of the temperature, we obtain, with an accuracy within an arbitrary constant

$$\mathfrak{s} = \mu e_{ik}{}^2 + \left(\frac{\lambda}{2}\, e_{ik}\delta_{ik} + \beta T)\right) e + C\theta$$

From the law of conservation of energy and from relations (11) we obtain the increment of heat

$$dh = \beta\,(T + \theta)\, de + C d\theta \tag{12}$$

Linearizing (12) and integrating with initial conditions $h = 0$, $e = 0$, $\theta = 0$ for $\tau = 0$ we have

$$h = \beta T e + C\theta \tag{13}$$

Thus, in accordance with (4) and (13), the equation relating the strains and the temperature of the elastic body is in the form of a generalized equation of heat conduction

$$C\,\frac{d\theta}{d\tau} + \beta T\,\frac{de}{d\tau} = k\nabla^2\theta + \frac{dw}{d\tau} \tag{14}$$

Equations (3) and (14), for given initial and boundary conditions, permit the determination of the temperature θ and the displacement vector $\mathbf{u}$ as a function of time and of the space coordinates.

Following Biot [1], we express the equations of thermoelasticity with the aid of two independent vectors, namely the displacement vector $\mathbf{u}$ and the vector $\mathbf{S}$, related to the temperature θ in accordance with

$$\frac{d\mathbf{S}}{d\tau} = -\frac{k}{T}\,\mathrm{grad}\,\theta \tag{15}$$

Eliminating with the aid of (15) the temperature on the right-hand side of the equation of heat conduction and carrying out the integration, we find

$$C\theta + \beta T e = - T \operatorname{div} \mathrm{S} + w + C_1 (x_k) \qquad \left(w = \int_0^\tau \frac{dw}{d\tau} d\tau\right) \tag{16}$$

Using the initial condition $\theta = 0$, $e = 0$, $\mathbf{S} = 0$, $w = 0$ for $\tau = 0$ we obtain $C_1(x_k) = 0$.

From Equation (16) it follows that the temperature may be determined as a function of two independent vectors $\mathbf{u}$, $\mathbf{S}$ and of heat supplied by sources

$$\theta = - \frac{T}{C} \left[\operatorname{div} (\mathrm{S} + \beta \mathbf{u}) - \frac{w}{T} \right] \tag{17}$$

The equation of heat conduction (14) may be transformed with the aid of (15) and (17) to the form

$$\frac{T}{k} \frac{d\mathrm{S}}{d\tau} + \operatorname{grad} \theta = 0 \tag{18}$$

To obtain the variational equations of thermoelasticity we multiply the equilibrium and the boundary conditions (3), as well as the equation of heat conduction (18), by the independent variations $\delta\mathbf{u}$ and $\delta\mathbf{S}$, respectively. Integrating over the volume and the surface we find

$$\begin{gathered} \iiint\limits_{(v)} [\mu \nabla^2 \mathbf{u} + (\lambda + \mu) \operatorname{grad} e - \beta \operatorname{grad} \theta] \, \delta \mathbf{u} \, dv = 0 \\ \iiint\limits_{(v)} \left(\frac{T}{k} \frac{d\mathrm{S}}{d\tau} + \operatorname{grad} \theta \right) \delta \mathrm{S} \, dv = 0, \qquad \iint\limits_{(\Omega)} (\sigma_{ik} l_k - P_i) \, \delta u_i d\Omega = 0 \end{gathered} \tag{19}$$

From relations (19) the identity follows

$$\begin{aligned} \iint\limits_{(\Omega)} (\sigma_{ik} l_k - P_i) \, \delta u_i \, d\Omega - \iiint\limits_{(v)} [\mu \nabla^2 \mathbf{u} + (\lambda + \mu) \operatorname{grad} e - \beta \operatorname{grad} \theta] \, \delta \mathbf{u} \, dv + \\ + \iiint\limits_{(v)} \left(\frac{T}{k} \frac{d\mathrm{S}}{d\tau} + \operatorname{grad} \theta \right) \delta \mathrm{S} \, dv = 0 \end{aligned} \tag{20}$$

Using the equation of Duhamel-Neumann (1), as well as the relation between the strains and the displacements, we transform the identity (20) to the form

$$\iint\limits_{(\Omega)} (\sigma_{ik} l_k - P_i) \, \delta u_i \, d\Omega - \iiint\limits_{(v)} \frac{\partial \sigma_{ik}}{\partial x_k} \delta u_i dv + \iiint\limits_{(v)} \left(\frac{T}{k} \frac{d\mathrm{S}}{d\tau} + \operatorname{grad} \theta \right) \delta \mathrm{S} \, dv = 0 \tag{21}$$

Transforming the triple integrals in Expression (21) by the formula of Gauss-Ostrogradski and taking into account the variation of the temperature in accordance with (17)

$$\delta\theta = -\frac{T}{c}\left[\operatorname{div}(\delta \mathbf{S} + \beta\delta\mathbf{u}) - \frac{\delta w}{T}\right] \tag{22}$$

we obtain the variational equation of thermoelasticity with heat sources and sinks in the form

$$\iiint\limits_{(v)}\left[\delta\left(W + \frac{C\theta^2}{2T}\right) - \frac{\theta}{T}\,\delta w\right]dv + \iiint\limits_{(v)} \frac{T}{k}\,\frac{d\mathbf{S}}{d\tau}\,\delta\mathbf{S}\,dv = \iint\limits_{(\Omega)} (\mathbf{P}\cdot\delta\mathbf{u} + \theta\mathbf{n}\cdot\delta\mathbf{S})\,d\Omega \tag{23}$$

Here W is the specific potential energy of isothermal deformation $(\theta = 0)$

$$W = \mu e_{ik}^2 + \frac{\lambda}{2}\, e_{ik} e \delta_{ik}$$

$\mathbf{p}$ is the vector of intensity of the surface loading, $\mathbf{n} = -(l_k \mathbf{j}_k)$ is the unit vector of the internal normal, $\mathbf{j}_k$ are the base vectors of the coordinate system x_k.

For the case when the heat sources and sinks are given functions of coordinates and time, the variational equation (23) coincides in form with the analogous equation of Biot [1]

$$\delta V + \iiint\limits_{(v)} \frac{T}{k}\,\frac{d\mathbf{S}}{d\tau}\,\delta\mathbf{S}\,dv = \iint\limits_{(\Omega)} (\mathbf{P}\delta\mathbf{u} + \theta\mathbf{n}\,\delta\mathbf{S})\,d\Omega \qquad \left(V = \iiint\limits_{(v)}\left(W + \frac{C\theta^2}{2T}\right)dv\right) \tag{24}$$

In contrast to Biot's equation, in calculating the variations by Formula (24) one should take into account the dependence of the temperature not only on the vectors $\mathbf{u}$, $\mathbf{S}$, but also on the sources of heat $w(x_k, \tau)$ in accordance with (17).

For independent $\mathbf{u}$, $\mathbf{S}$ and w, the variation with respect to $\mathbf{u}$ gives the equation of equilibrium and the force boundary conditions (3), while variation with respect to $\mathbf{S}$ yields the equation of heat conduction (14).

Let us note that for independent $\mathbf{u}$, $\mathbf{S}$ and given distribution of heat sources and sinks ($\delta w = 0$) the variational equation (23) may be written down in the form of Lagrange's equations for a system with energy dissipation:

$$\frac{\partial V}{\partial q_n} + \frac{\partial D}{\partial q_n} = Q_n \qquad \left(D = \iiint\limits_{(v)} \frac{T}{2k}\left(\frac{\partial \mathbf{S}}{\partial \tau}\right)^2 dv\right) \tag{25}$$

D plays here the role of the dissipation function, and V that of the potential energy of the system; q_n are the generalized coordinates which may be used to express the vectors

$$\mathbf{u} = \sum_{n=1}^{\infty} q_n(\tau)\,\mathbf{u}_n(x_k), \qquad \mathbf{S} = \sum_{n=1}^{\infty} q_n(\tau)\,\mathbf{S}_n(x_k)$$

as well as the quantities V and D. The corresponding generalized forces are

$$Q_n = \iint\limits_{(\Omega)} \left(\mathbf{P}\, \frac{\partial \mathbf{u}}{\partial q_n} + \theta \mathbf{n}\, \frac{\partial \mathbf{S}}{\partial q_n} \right) d\Omega$$

Let us show under what conditions the variational equation of thermoelasticity (23), which takes into account irreversible phenomena in thermoelastic processes, may be replaced by variational equations of classical thermodynamics of equilibrium processes.

Eliminating $d\mathbf{S}/d\tau$ from the basic equation (23) with the aid of (18) and integrating by parts, we obtain

$$\delta V - \iiint\limits_{(v)} \frac{\theta}{T}\, \delta w\, dv + \iiint\limits_{(v)} \theta \operatorname{div} \delta\mathbf{S} dv = \iint\limits_{(\Omega)} \mathbf{P} \cdot \delta\mathbf{u} d\Omega \tag{26}$$

Replacing in (26) the generalized free energy by means of the internal energy

$$V = \iiint\limits_{(v)} \left[\partial - \left(C\theta + \beta T e - \frac{c\theta^2}{2T} \right) \right] dv$$

and linearizing in accordance with (12), (13), we find

$$\delta \partial - \iiint\limits_{(v)} (\beta T \delta e + C \delta\theta)\, dv = \iint\limits_{(\Omega)} \mathbf{P} \delta\mathbf{u} d\Omega \tag{27}$$

Here ∂ is the internal energy of the system. From (27) we obtain on the basis of (13)

$$\delta \partial - \iiint\limits_{(v)} \delta h\, dv = \iint\limits_{(\Omega)} \mathbf{P} \cdot \delta \mathbf{u}\, d\Omega \tag{28}$$

In the particular case of thermoelastic processes which are not accompanied by a supply or removal of heat, the variation δh should be set equal to zero and Equation (28), following from the basic equation of thermoelasticity (23), passes into the variational equation of adiabatic processes of classical thermodynamics.

One should keep in mind that the variations $\delta\theta$ and δe for an adiabatic process ($\delta h = 0$), in accordance with (13), are related by the additional expression

$$\delta\theta = -\frac{\beta T}{C}\, \delta e$$

Let us now transform the basic variational equation (23) for the case of an isothermal process. Using (26) and replacing $\operatorname{div} \delta\mathbf{S}$ with the help of (17)

$$\mathrm{div}\,\delta\mathbf{S} = -\frac{C}{T}\delta\theta - \beta\delta e + \frac{1}{T}\delta w \tag{29}$$

we find

$$\delta\iiint_{(v)}(W - \beta\theta e)\,dv + \iiint_{(v)}\beta e\delta\theta dv = \iint_{(\Omega)}\mathbf{P}\cdot\delta\mathbf{u}d\Omega \tag{30}$$

The variation of the temperature $\delta\theta$, on the basis of the definition of the vector $\mathbf{S}$ and relation (29), may be represented in the form

$$\delta\theta = \frac{T}{C}\delta\left(\frac{w}{T} - \beta e + \frac{k}{T}\int_0^\tau \nabla^2\theta d\tau\right) \tag{31}$$

From Expression (31) it follows that an isothermal process of deformation of a non-uniformly heated body ($\delta\theta = 0$) is realized only for a stationary temperature field, when

$$\nabla^2\theta = 0 \tag{32}$$

and in the presence of a special system of distribution of sources and sinks

$$w = \beta Te \tag{33}$$

compensating the heat produced in the elastic body during the process of deformation.

In this case the variation $\delta\theta$ in Expression (30) may be set equal to zero, and the variational equation of an isothermal process takes on the form

$$\delta F = \iint_{(\Omega)}\mathbf{P}\cdot\delta\mathbf{u}d\Omega \tag{34}$$

Here F is the classical free energy (Helmholtz's potential)

$$F = \iiint_{(V)}(W - \beta\theta e)\,dv$$

The variational equation (34) is identical with the equation of thermoelasticity of classical thermodynamics of equilibrium processes in the form suggested by Hemp [3], or if one introduces some fictitious body and surface forces

$$\mathbf{N}_v = -\beta\,\mathrm{grad}\,\theta, \qquad \mathbf{N}_\Omega = -\beta\theta\mathbf{n}$$

to the variational equation of thermoelasticity due to Kachanov [2]

$$\delta\iiint_{(v)}Wdv = \iint_{(\Omega)}(\mathbf{P} - \beta\theta\mathbf{n})\,\delta\mathbf{u}d\Omega - \iiint_{(v)}\beta\,\mathrm{grad}\,\theta\delta\mathbf{u}dv$$

Thus, the variational equations of Kachanov and Hemp which correspond to the isothermal problem of thermoelasticity are valid, strictly speaking, only for a stationary temperature field ($\nabla^2\theta = 0$) for sources and sinks $w = \beta Te$.

These variational equations may be considered as approximate if during the process of deformation the phenomenon of heat conduction may be neglected.

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